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THE FUNDAMENTAL SOLUTION OF THE SPACE-TIME FRACTIONAL DIFFUSION EQUATION

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Dedicated to Rudolf Gorenflo,
Prof. Emeritus of the Free University of Berlin,
on the occasion of his 70-th birthday (July 30, 2000)

Abstract

We deal with the Cauchy problem for the space-time fractional diffusion equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0,2]$ and skewness θ ($|\theta| \leq \min\{\alpha, 2-\alpha\}$), and the first-order time derivative with a Caputo derivative of order $\beta \in (0,2]$. The fundamental solution (Green function) for the Cauchy problem is investigated with respect to its scaling and similarity properties, starting from its Fourier-Laplace representation. We review the particular cases of space-fractional diffusion $\{0 < \alpha \leq 2, \beta = 1\}$, time-fractional diffusion $\{\alpha = 2, 0 < \beta \leq 2\}$, and neutral-fractional diffusion $\{0 < \alpha = \beta \leq 2\}$, for which the fundamental solution can be interpreted as a *spatial probability density function evolving in time.* Then, by using the Mellin transform, we provide a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane, which allows us *to extend the probability interpretation to the ranges* $\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$ *and* $\{1 < \beta \leq \alpha \leq 2\}$. Furthermore, from this representation we derive explicit formulae (convergent series and asymptotic expansions), which enable us to plot the spatial probability densities for different values of the relevant parameters α, θ, β.

Mathematics Subject Classification: 26A33, 33E12, 33C40, 44A10, 45K05, 60G52

Key Words and Phrases: diffusion, fractional derivative, Fourier transform, Laplace transform, Mellin transform, Mittag-Leffler function, Wright function, Mellin-Barnes integrals, Green function, stable probability distributions